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Chaotic behaviour and order to chaos transition of 't Hooft–Polyakov monopoles

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Abstract. Chaotic behaviour of time-dependent spherically symmetric SU(2) Yang–Mills–Higgs system near the 't Hooft–Polyakov monopole solution is investigated by calculating the maximal Lyapunov exponents. As a parameter depending on the self-interaction constant of scalar fields increases there is a phase-transition-like behaviour from order to chaos in the system. Presence of Higgs scalar fields reduces chaos in the Yang–Mills system.

1. Introduction

Recently much interest has been focused on the question of non-integrability and chaos in classical non-Abelian gauge theories. The spatially homogeneous Yang–Mills system (YMCM) is non-integrable and shows strong chaotic properties in general. This has been established by many authors using various analytical and numerical techniques such as the study of the instability of periodic solutions, Poincaré sections, calculation of Lyapunov exponents, singular point analysis, etc (Matinyan *et al* 1981a, Nikolaevskii and Schur 1982, 1983, Asatryan and Savvidy 1983, Savvidy 1984, Gorsky 1984, Steeb *et al* 1986, Furusawa 1987, Villarroel 1988, Karkowski 1990.) Studies on the more important and more realistic spacetime-dependent systems are much less in number. Studies on such non-Abelian field theoretic systems are of relevance in understanding quark confinement in QCD, monopole stability, etc. Study of spatio-temporal chaos in itself is also very interesting. Matinyan *et al* (1986, 1988) showed that the space-time-dependent Yang–Mills system can also exhibit dynamical chaos. They studied time-dependent spherically symmetric solutions of the SU(2) Yang–Mills system, in particular the Wu–Yang monopole solution. Exponential instability of trajectories was found using the Fermi–Pasta–Ulam technique of studying the distribution of energy among different harmonic modes. Kawabe and Ohta (1990) studied the system further by calculating the induction period, the equal time correlation and the maximal Lyapunov exponents and showed the existence of chaos in the YM system. Using the technique of Painlevé analysis Joy and Sabir (1989) have recently shown that time-dependent spherically symmetric SU(2) Yang–Mills and Yang–Mills–Higgs systems are non-integrable.

Chaotic behaviour of classical systems with spontaneous symmetry breaking is also very interesting and first investigations were made by Matinyan *et al* (1981b). They found an order to chaos transition in the spatially uniform Yang–Mills system with Higgs scalar fields (YMHCM), as the vacuum expectation value of Higgs field is changed. Recently Matinyan *et al* (1989) performed some preliminary numerical calculations on time dependent spherically symmetric SU(2) Yang–Mills–Higgs system (SSYMH) and showed that there can be chaos. Details of chaotic behaviour of SSYMH is unclear and whether there is an order to chaos transition similar to YMHCM is an open question.

In this paper we present the results of a numerical study on the chaotic behaviour of the SSYMH system. We consider specifically the 't Hooft-Polyakov monopole solution. Because of the large mass of the monopole, quantum fluctuations are reduced and the classical system may be a good approximation to the real quantum case. We find a phase-transition-like behaviour from order to chaos as we tune the parameter which depends on the self-interaction constant of scalar fields. For our study we discretize the system into a collection of interacting coupled nonlinear oscillators and calculate the maximal Lyapunov exponents for various parameter values and different number of oscillators. Calculation of maximal Lyapunov exponents is a reliable criterion to determine whether a system is chaotic or not.

In the next section we briefly describe the system under investigation. We present the numerical techniques applied and results in section 3. Section 4 contains our conclusions.

2. Yang–Mills–Higgs system: 't Hooft–Polyakov monopole

't Hooft (1974) and Polyakov (1974) discovered magnetic monopoles as finite energy solutions of non-Abelian gauge theories. They considered the Georgi-Glashow model with gauge group SU(2) broken down to U(1) by Higgs triplets. The Lagrangian of the model is

$$\mathscr{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \tfrac{1}{2}D_\mu\phi^a D^\mu\phi^a - V(\phi) \tag{1}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon_{abc}A^b_\mu A^c_\nu$$
$$D_\mu\phi_a = \partial_\mu\phi_a + g\varepsilon_{abc}A^b_\mu\phi_c$$

and

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2.$$

The equations of motion are

$$\begin{aligned} D_\nu F^{\mu\nu a} &= -g\varepsilon_{abc}(D^\mu\phi_b)\phi_c \\ D_\mu D^\mu\phi_a &= (m^2 - \lambda\phi^2)\phi_a. \end{aligned} \tag{2}$$

The vacuum expectation value of the scalar field and Higgs boson mass are $\langle\phi^2\rangle = F^2 = m^2/\lambda$ and $M_H = \sqrt{2\lambda}\, F$ respectively. Mass of the gauge boson is $M_w = gF$. Using the time-dependent 't Hooft-Polyakov ansatz (Mecklenberg and O'Brien 1978):

$$\begin{aligned} &A^a_0 = 0 \qquad A^a_i = -\varepsilon_{ain}r_n\frac{[1-K(r,t)]}{r^2} \\ &\phi_a = \frac{1}{g}r_a\frac{H(r,t)}{r^2} \end{aligned} \tag{3}$$

where $r_n = x_n$ and r is the radial variable, the field equations (2) can be written as

$$\begin{aligned} r^2(\partial_r^2 - \partial_t^2)K &= K(K^2 + H^2 - 1) \\ r^2(\partial_r^2 - \partial_t^2)H &= H\left(2K^2 - m^2r^2 + \frac{\lambda H^2}{g^2}\right). \end{aligned} \tag{4}$$

With $\beta=\lambda/g^2=M_H^2/2M_w^2$ and introducing the variables $\xi=M_w r$ and $\tau=M_w t$, the equations (4) become

$$\begin{aligned}(\partial_\xi^2-\partial_\tau^2)K&=K(K^2+H^2-1)/\xi^2\\(\partial_\xi^2-\partial_\tau^2)H&=H(2K^2+\beta(H^2-\xi^2))/\xi^2.\end{aligned}\tag{5}$$

Total energy of the system E is given by

$$\begin{aligned}C(\beta)&=\frac{g^2E}{4\pi M_w}\\&=\int_0^\infty\left\{K_\tau^2+\frac{H_\tau^2}{2}+K_\xi^2+\frac{1}{2}\left(H_\xi-\frac{H}{\xi}\right)^2+\frac{1}{2\xi^2}(K^2-1)^2\right.\\&\qquad\left.+\frac{K^2H^2}{\xi^2}+\frac{\beta}{4\xi^2}(H^2-\xi^2)^2\right\}\mathrm{d}\xi.\end{aligned}\tag{6}$$

A time-independent version of the ansatz (3) gives the 't Hooft–Polyakov monopole solution with winding number 1. For finiteness of energy the field variables should satisfy the following conditions,

$$H\to0\qquad K\to1\quad\text{as}\quad\xi\to0$$

and

$$H\to\xi\qquad K\to0\quad\text{as}\quad\xi\to\infty.$$

The 't Hooft-Polyakov monopole is more realistic than the Wu–Yang monopole; it is non-singular and has finite energy. In the limit $\beta\to0$, known as the Prasad–Sommerfeld (PS) limit, we have the static solutions,

$$\begin{aligned}K(\xi)&=\xi/\sinh\xi\\H(\xi)&=\xi\coth\xi-1.\end{aligned}$$

It has not been possible to find exact non-trivial solutions for $\beta\neq0$ analytically.

3. Lyapunov exponents and order to chaos transition

Non-integrable dynamical systems may show irregular, random, or unpredictable behaviour known as chaos. To study chaos there exist many techniques, mostly numerical. Chaos is characterized by a sensitive dependence on initial conditions, nearby trajectories diverging exponentially. One quantitative measure of chaos is the magnitude of Lyapunov exponents (LE), which are the average rate of exponential divergence of nearby trajectories. Calculation of LE is a reliable and convenient way to study chaos. If the maximal LE is greater than zero the system is said to be chaotic. Maximal LE can be calculated in the following way.

Let

$$\dot{x}_i=F_i(\boldsymbol{x})\qquad i=1,\ldots,n\tag{7}$$

where n is the dimension of the system and $\boldsymbol{x}$ is an n-vector, be a general finite dimensional dynamical system. Here the over-dot denotes time derivative. The corresponding linearized variational system is given by

$$\dot{y}_i = \sum_j^n \frac{\partial F_i(x)}{\partial x_j} y_j. \tag{8}$$

One-dimensional LE are defined as (Oseledec 1968)

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln|y(t)| \tag{9}$$

where $|.|$ indicates the norm. If one chooses the initial variations $y_i(0)$ at random, one obtains maximal LE. Since for a chaotic system $|y(t)|$ increases exponentially with time, in numerical calculations we have to renormalize it at a suitable time step (Lichtenberg and Lieberman 1983).

For our study we discretize the original infinite dimensional system (5) to obtain a set of N coupled anharmonic oscillators. The discrete model is given by

$$\begin{aligned}\ddot{K}(i,t) &= \frac{K(i+1,t)-2K(i,t)+K(i-1,t)}{h^2} - \frac{K(i,t)[K(i,t)^2+H(i,t)^2-1]}{(ih)^2}\\ \ddot{H}(i,t) &= \frac{H(i+1,t)-2H(i,t)+H(i-1,t)}{h^2}\\ &\quad - \frac{2H(i,t)K(i,t)^2-\beta H(i,t)[H(i,t)^2-(ih)^2]}{(ih)^2} \qquad i=1,\ldots,N-1\end{aligned} \tag{10}$$

where h is the space discretization step. Corresponding variational system is obtained by discretizing the following equations:

$$\begin{aligned}(\partial_\xi^2-\partial_\tau^2)\delta K &= \frac{(3K^2+H^2-1)\delta K+2HK\delta H}{\xi^2}\\ (\partial_\xi^2-\partial_\tau^2)\delta H &= \frac{(2K^2+3\beta H^2-\beta\xi^2)\delta H+4KH\delta K}{\xi^2}.\end{aligned} \tag{11}$$

For calculating LE we have to solve system (10) along with the variational system obtained from (11). In the system, there exist two parameters, the energy and the value of β. For the numerical integration we can start from arbitrary values of K and H. But we are interested in the evolution of 't Hooft–Polyakov monopole solutions. Static monopole solutions occur at the minimum of the energy functional $C(\beta)$ for a fixed β, so we choose $K(i,0)$ and $H(i,0)$ as the static solution of the YMH system, which we find using a finite difference method for solving boundary value problems. We use the asymptotic form of the solutions for fixing the boundary values. $C(\beta)$ for different β values are given in table 1. Static solutions of SSYMH for some β values are shown in figures 1(a) and 1(b).

We use fixed boundary conditions and numerically solve the system (10) with static solutions as initial conditions along with the discretized system obtained from (11). For our calculations we take $N=100$ and the discretization step $h=0.1$. In figures 2(a) and 2(b) plots of LE versus time are given for some values of β. We calculate up to $t=1000.0$, which is sufficient for obtaining asymptotic values of LE. We used an IMSL routine for the Bulirsh–Stoer algorithm for numerical integration of the differential

Table 1. $C(\beta)$ and maximal LE for different β values.

β	$C(\beta)$	LE
0.0	1.000	0.00
0.1	1.006	0.00
0.5	1.193	0.00
1.0	1.243	0.00
2.0	1.302	0.00
5.0	1.386	0.00
10.0	1.451	0.00
50.0	1.600	0.00
75.0	1.641	1.54E-3
100.0	1.671	2.32E-3
200.0	1.762	1.13E-2
500.0	1.971	2.54E-2
1000.0	2.301	7.01E-2
5000.0	4.641	1.00E-1

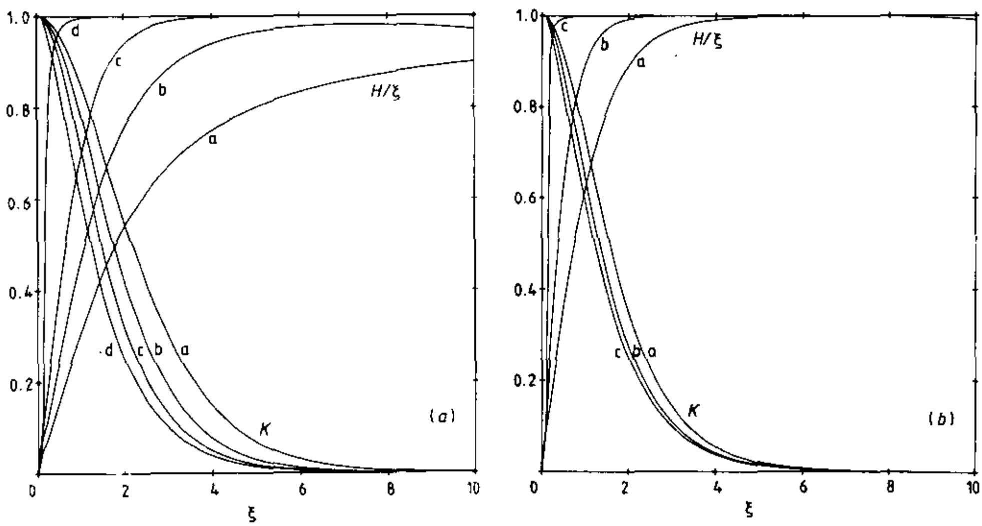

Figure 1. K and H/ξ versus ξ. (*a*) For (a) $\beta=0.0$, (b) $\beta=0.1$, (c) $\beta=1.0$, and (d) $\beta=100.0$. (*b*) For (a) $\beta=0.5$, (b) $\beta=5.0$, and (c) $\beta=500.0$.

equations with a tolerance value 10^{-3}. Calculations are done in double precision in a CYBER 180/830 computer. In the case of $\beta=1000$ and $\beta=5000$ we used a higher tolerance value because of the enormous amount of computer time required otherwise. We did the calculations with high accuracy such that change in energy is less than 1%. Lyapunov exponents for different values of β are also given in table 1. From figure 3, where LE versus $\log(\beta)$ is plotted, we can see that there is a transition from order to chaos near $\beta=75.0$. Up to $\beta=50.0$ LE is zero within the limits of numerical accuracy. For $\beta=75.0$ LE becomes positive and reaches an asymptotic value 1.2×10^{-3}. For higher β values we get higher and higher positive LEs. However LE is not seen increasing indefinitely with β. At the transition region the increase is rapid but as β increases further the rate of increase in LE falls. As $\beta\to\infty$, LE appears to attain an asymptotic value.

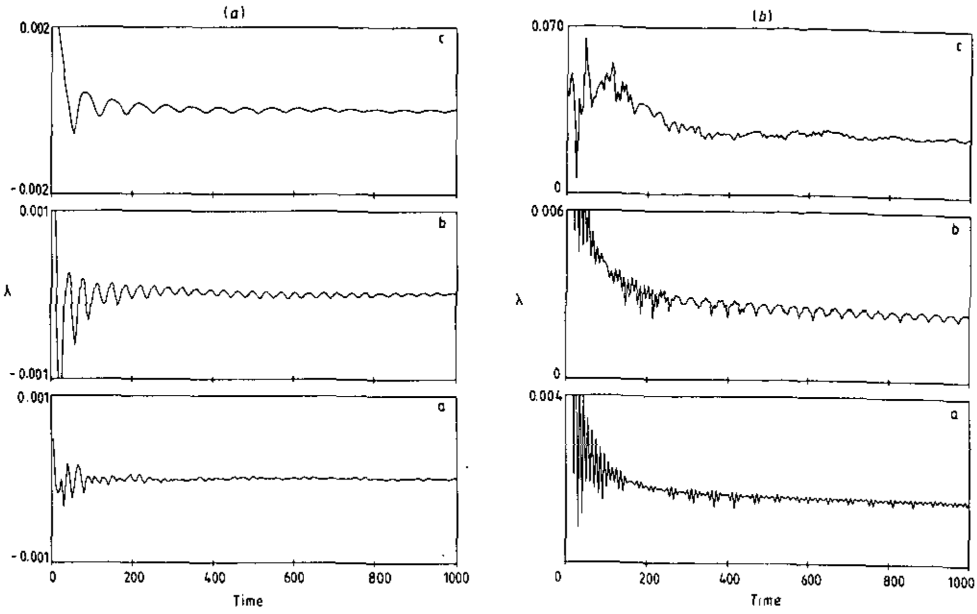

Figure 2. Maximal LE (λ) versus time. (*a*) For (a) $\beta = 0.0$, (b) $\beta = 10.0$, and (c) $\beta = 50.0$. (*b*) for (a) $\beta = 75.0$, (*b*) $\beta = 100.0$, and (c) $\beta = 500.0$.

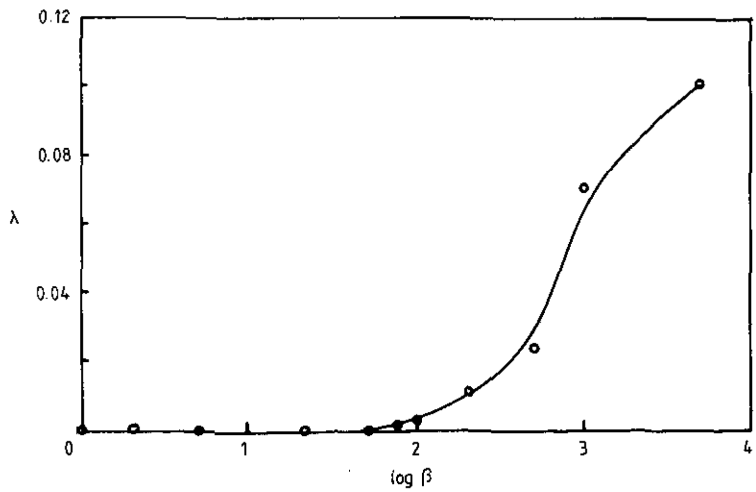

Figure 3. Maximal LE (λ) versus log(β).

We have repeated the calculations with different values of N also. Results are qualitatively the same as those for $N = 100$. LE for $N = 16, 32, 64, 100$ are given for some β values in table 2. Increasing N does not have much effect beyond $N = 32$. This indicates that the results obtained are good approximations to the original infinite dimensional system. This behaviour can be compared to the results obtained by Livi *et al* (1986) for a Fermi–Pasta–Ulam chain of anharmonic oscillators. For the FPU β-model LE reaches an asymptotic value when the number of oscillators is $N = 20$–40. For small N values boundary values also have effects on the dynamics. Asymptotic

Table 2. Maximal LE for different values of N and β.

	LE			
β	$N=16$	32	64	100
50.0	0.00	0.00	0.00	0.00
75.0	0.006	0.0012	0.0013	0.0015
100.0	0.0015	0.0021	0.0021	0.0023
200.0	0.027	0.0091	0.0105	0.011

values of K and H are reached only after $\xi = 3$–4. The quartic oscillator system corresponding to $N = 1$ is non-integrable and chaotic for all β values.

4. Conclusions

Our calculations show that there is a phase-transition-like behaviour from order to chaos in the SU(2) SSYMH system. This result is in agreement with that obtained in the case of a spatially homogeneous YMH system, where Higgs field manifests only as the vacuum expectation value F. As F increases there is an order to chaos transition and in that case there are no terms dependent on the self-interaction constant. There is only one parameter for YMHCM, namely $g^2E/4\pi M_w$. On the other hand, here we considered the time evolution of both gauge and scalar fields and there exist two parameters $C(\beta)$ and β. β depends on the self-interaction constant λ. Since we are interested in monopole solutions we took the minimum value of energy functional $C(\beta)$ for a specific β value. It is known that as β increases the effect of Higgs field decreases and when $\beta \to \infty$ the system becomes purely Yang–Mills, which is highly chaotic. The effect of Higgs scalar fields is to reduce the stochasticity of the YM system. In the central part of the monopole the scalar field is approximately equal to zero and the YM field which dominates this region displays chaotic behaviour. Outside the monopole core the Higgs field approaches its mean value and the YM field behaves in a regular manner. From our study one can see that 't Hooft–Polyakov monopole solutions show irregular behaviour in time, and they are exponentially unstable. Our results can be compared with those of Brandt and Neri (1979) in the context of Wu–Yang monopoles. They have shown that negative modes exist in the spectrum of the operator describing small perturbations of monopole solutions, implying exponential growth of perturbations with time. Solutions with magnetic charge $q \geq 1$ are unstable. Arbitrary continuous deformations of the field configurations do not change the topological charge during the evolution of the fields with time. The evolution of the fields in the central part of the monopole can be arbitrarily complicated and may oscillate or vary ergodically. Though in the case of the monopole classical description may be a meaningful approximation to the quantum case the implications of the result in the exact quantum field theory of this object is a separate issue requiring detailed study.

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